DETERMINATION OF HEAT TRANSFER COEFFICIENT FROM TEMPERATURE MEASUREMENTS DURING HEATING OF AN INFINITE PLATE

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A method is described for determining the heat transfer coefficient from measured temperatures in the medium and on a heated flat plate.

During heating of an infinite plate which has one surface thermally insulated and the other experiencing heat exchange with the surrounding medium, the heat transfer coefficient can be determined only from analysis of the dynamics of variation of the temperature field.

For the sake of generality we shall assume that the heat transfer coefficient is not constant during heating, but is, for example, a function of time. We shall take the specific heat and the thermal conductivity of the plate material to be constant.

Under these assumptions the heat conduction equation for an infinite plate and the boundary conditions will be as follows:

$$\frac{\partial \Theta (X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 \Theta (X,\ \mathrm{Fo})}{\partial X^2}, \tag{1}$$

$$\frac{\partial \Theta (1,\ \mathrm{Fo})}{\partial X} = \mathrm{Bi}\,(\mathrm{Fo})\,[\Theta_c\,(\mathrm{Fo}) - \Theta\,(1,\ \mathrm{Fo})], \tag{2}$$

$$\frac{\partial \Theta (0,\ \mathrm{Fo})}{\partial X} = 0. \tag{3}$$

We shall assume that the initial plate temperature distribution is, in general, nonuniform:

$$\Theta\,(X,\ 0) = f\,(X). \tag{4}$$

In the system of equations (1)-(4) we have to determine the function Bi(Fo) from measurement of temperatures in the medium and on the plate at one or several points across it during the heating.

The problem can be simplified by varying the medium temperature according to some previously determined law. For example, we can maintain unchanged the temperature difference between the heated and insulated plate surfaces. But, in practice, one cannot maintain such conditions.

The present paper describes a method for determining Bi(Fo) from measured temperatures of the medium and the heated surface during heating of the plate, assuming an arbitrary variation in the medium temperature and an arbitrary continuous initial temperature distribution, Eq. (4).

Then, knowing $\Theta_c(\mathrm{Fo})$ and $\Theta(1, \mathrm{Fo})$ the function Bi(Fo) can easily be found from Eq. (2) if we know the relation between $\partial\Theta(1, \mathrm{Fo})/\partial X$ and $\Theta(1, \mathrm{Fo})$. To determine the latter, we make use of the fact that the manner in which the derivative with respect to X at the point X = 1 depends on the surface temperature is invariant with respect to the boundary conditions on that surface, if the initial and final conditions are unchanged on the other surface. In other words, for any boundary conditions on the surface X = 1 and the identical initial and boundary conditions at X = 0, this dependence will remain unchanged, i.e., for unchanged boundary conditions at X = 1 the derivative with respect to X and the temperature at the point X = 1 will always vary with

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time so that they satisfy the same relation. In the special case of boundary conditions of the third kind, the invariance means that the relation considered will remain unchanged for any function Bi(Fo), and can be found, in particular, by solving Eq. (1) with conditions (3) and (4) and boundary conditions of the first kind on the surface X = 1:

$$\Theta(1,\ \mathrm{Fo}) = \varphi(\mathrm{Fo}). \tag{2'}$$

(The proof of this is given in the Appendix).

In what follows we shall understand $\varphi(\mathrm{Fo})$ in Eq. (2') to be the temperature of the heated surface measured during heating of the plate.

We can represent the solution of the system (1), (2'), (3), and (4), because of its linearity, in the form of the sum of two functions

$$\Theta(X,\ \mathrm{Fo}) = \Theta_1(X,\ \mathrm{Fo}) + \Theta_2(X,\ \mathrm{Fo}), \tag{5}$$

where $\Theta_1(X, \mathrm{Fo})$ is a solution of the above system with zero initial conditions, and $\Theta_2(X, \mathrm{Fo})$ is its solution at zero temperature $\Theta(1, \mathrm{Fo})$.

Differentiating Eq. (5) with respect to X and substituting the value X = 1 we obtain

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} = \frac{\partial\Theta_1(1,\ \mathrm{Fo})}{\partial X} + \frac{\partial\Theta_2(1,\ \mathrm{Fo})}{\partial X}. \tag{6}$$

We shall find each term in Eq. (6) as a function of $\varphi(\mathrm{Fo})$ and f(X).

Let the function H(X, Fo) describe the temperature field of the plate for a discontinuous unit change $\Theta(1, \mathrm{Fo})$ and zero initial conditions. Then H(X, Fo) is given by the formula [1, 2]:

$$H(X,\ \mathrm{Fo}) = 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\nu_n}\cos\nu_n X\exp(-\nu_n^2\mathrm{Fo}), \tag{7}$$

in which

$$\nu_n = \frac{2n-1}{2}\pi,\ n = 1,\ 2,\ \ldots$$

Differentiating Eq. (7) with respect to X, we obtain the expression

$$\frac{\partial H(X,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin\nu_n X\exp(-\nu_n^2\mathrm{Fo}). \tag{8}$$

We substitute the value X = 1 into Eq. (8), and, using the equality

$$\sin\nu_n = (-1)^{n+1},\quad n = 1,\ 2,\ \ldots,$$

we reduce it to the form

$$\frac{\partial H(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty}\exp(-\nu_n^2\mathrm{Fo}). \tag{9}$$

The series in Eq. (9) converges uniformly in the interval $(0, \infty)$, and, therefore, for Fo > 0 it determines the transfer function with respect to $\partial\Theta_1(1, \mathrm{Fo})/\partial X$ upon perturbation of the temperature $\Theta(1, \mathrm{Fo})$.

According to the Duhamel theorem, and taking into account Eq. (9), we can write the desired relation in the form

$$\frac{\partial\Theta_1(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty}\left\{\varphi(0) + \int_0^{\mathrm{Fo}}\frac{d\varphi(\tau)}{d\tau}\exp(\nu_n^2\tau)\,d\tau\right\}\exp(-\nu_n^2\mathrm{Fo}). \tag{10}$$

Here we postulate that $\varphi(\mathrm{Fo})$ is a continuous function, and that $\varphi(0)$ is its initial value.

We shall take into account the effect of the initial conditions (4). Substituting the function $\Theta_2(X, \mathrm{Fo})$ into Eq. (5), we can write it in the form [1, 2]

$$\Theta_2(X,\ \mathrm{Fo}) = 2\sum_{n=1}^{\infty} \cos\nu_n X \int_0^1 f(X)\cos\nu_n X\,dX \exp(-\nu_n^2\mathrm{Fo}). \tag{11}$$

Differentiating Eq. (11) with respect to X, and substituting the value X = 1, we obtain, for Fo > 0, the expression

$$\frac{\partial\Theta_2(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty}(-1)^n\nu_n\int_0^1 f(X)\cos\nu_n X\,dX \exp(-\nu_n^2\mathrm{Fo}). \tag{12}$$

We shall designate

$$z_n(0) = (-1)^n\nu_n\int_0^1 f(X)\cos\nu_n X\,dX,\quad n=1,\ 2,\ \ldots, \tag{13}$$

and rewrite Eq. (12) in the form

$$\frac{\partial\Theta_2(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty} z_n(0)\exp(-\nu_n^2\mathrm{Fo}).$$

We shall substitute Eqs. (10) and (13) into Eq. (5), and finally obtain the relation we seek

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty}\left\{[z_n(0)+\varphi(0)] + \int_0^{\mathrm{Fo}}\frac{d\varphi(\tau)}{d\tau}\exp(\nu_n^2\tau)\,d\tau\right\}\exp(-\nu_n^2\mathrm{Fo}), \tag{14}$$

or

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty} z_n(\mathrm{Fo}), \tag{15}$$

where

$$z_n(\mathrm{Fo}) = \left\{[z_n(0)+\varphi(0)] + \int_0^{\mathrm{Fo}}\dot{\varphi}(\tau)\exp(\nu_n^2\tau)\,d\tau\right\}\exp(-\nu_n^2\mathrm{Fo}),\quad n=1,\ 2,\ \ldots \tag{16}$$

Differentiating the left and right sides of Eq. (16) with respect to Fo, we obtain an infinite system of ordinary differential equations for determining $z_n(\mathrm{Fo})$, n = 1, 2,...,

$$\frac{1}{\nu_n^2}\dot{z}_n(\mathrm{Fo}) + z_n(\mathrm{Fo}) = \frac{1}{\nu_n^2}\dot{\varphi}(\mathrm{Fo}),\quad n=1,\ 2,\ \ldots, \tag{17}$$

with initial conditions $z_n(0)$, n = 1, 2,..., and $\varphi(0)$.

We shall designate

$$y_n(\mathrm{Fo}) = \varphi(\mathrm{Fo}) - z_n(\mathrm{Fo}),\quad n=1,\ 2,\ \ldots \tag{18}$$

Then, in the new variables, the system of equations takes the form

$$\frac{1}{\nu_n^2}\dot{y}_n(\mathrm{Fo}) + y_n(\mathrm{Fo}) = \varphi(\mathrm{Fo}),\quad n=1,\ 2,\ \ldots \tag{19}$$

It can be shown that the initial condition of the system (19) will be as follows:

$$y_n(0) = -z_n(0) = (-1)^{n+1}\nu_n\int_0^1 f(X)\cos\nu_n X\,dX,\quad n=1,\ 2,\ \ldots \tag{20}$$

It follows from Eqs. (15) and (18) that

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} = 2\sum_{n=1}^{\infty}[\varphi(\mathrm{Fo}) - y_n(\mathrm{Fo})]. \tag{21}$$

From the boundary condition (2), taking into account Eq. (21), we obtain a formula for determining the function Bi(Fo)

$$\mathrm{Bi}(\mathrm{Fo}) = \frac{2\sum_{n=1}^{\infty}[\varphi(\mathrm{Fo}) - y_n(\mathrm{Fo})]}{\Theta_c(\mathrm{Fo}) - \varphi(\mathrm{Fo})}. \tag{22}$$

Thus, determination of Bi(Fo) reduces, basically, to integration of Eq. (19) with initial conditions (20).

For a parabolic initial temperature distribution through the plate

$$f(X) = f(0) + [f(1) - f(0)]X^2 \tag{23}$$

the initial conditions (20) are computed from the formula

$$y_n(0) = f(1) - \frac{2}{\nu_n^2}[f(1) - f(0)], \quad n = 1,\ 2,\ \ldots \tag{24}$$

Since the function φ(Fo) is given graphically (it is the temperature of the heated surface as measured during heating of the plate), it is convenient to carry out graphical integration by the Bashkirov method, which is widely used to investigate control processes in automatic systems [3]. Since all the equations of Eq. (19) are of first order with constant coefficients, their integration by the above method is a simple operation.

The method does not impose any restriction on the form of the function φ(Fo).

Since the time constants in Eq. (19) decrease rapidly with increase of the number n ($1/\nu_1^2$ = 0.4053; $1/\nu_2^2$ = 0.0450; $1/\nu_3^2$ = 0.162; $1/\nu_4^2$ = 0.0083; $1/\nu_5^2$ = 0.0050; $1/\nu_6^2$ = 0.0063), we can restrict ourselves to solution of a finite number of these equations without much loss in accuracy. The required number of differential equations to be calculated is immediately evident upon integration. In practice, it is sufficient to take into account only 2-3 of the first equations of Eq. (19).

Figure 1 gives the results of determination of Bi, accounting for the first two equations of Eq. (19) with respect to the temperature Θ(1, Fo) = φ(Fo) of the heated plate surface when the plate heating results from a unit stepwise change in the medium temperature, Θ_c = 1, with zero initial conditions f(X) = 0 and Bi = 2, obtained by solving the system of equations (1)-(4). In determining Bi, the time instant Fo = 0.4 will be taken as the initial value. The temperature distribution through the wall at this time is approximated by a parabola. Figure 1 shows the solution obtained by the Bashkirov method, y_1(Fo) and y_2(Fo), of the two equations considered, and also shows the differences ($\Theta_c - \varphi$), ($\varphi - y_1$) and $\sum_{n=1}^{2}(\varphi - y_n)$. The fact that curves 1 and 3 are close allows us to determine the Biot number, assumed equal to 2.0 in constructing φ(Fo), with an accuracy of 4-5%, which is sufficient for engineering calculations.

APPENDIX

Let a function Θ(X, Fo) satisfy Eq. (1) with boundary conditions (2)-(4), and let W(X, Fo) be a solution of the equation

$$\frac{\partial W(X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 W(X,\ \mathrm{Fo})}{\partial X^2} \tag{A.1}$$

with conditions

$$W(1,\ \mathrm{Fo}) = \Theta(1,\ \mathrm{Fo}), \tag{A.2}$$

$$\frac{\partial W(1,\ \mathrm{Fo})}{\partial X} = 0, \tag{A.3}$$

$$W(X,\ 0) = f(X). \tag{A.4}$$

The function Θ(X, Fo) also evidently satisfies the system (A.1)-(A.4). Because of the uniqueness of the solution of the system, we conclude that

$$\Theta(X,\ \mathrm{Fo}) \equiv W(X,\ \mathrm{Fo}). \tag{A.5}$$

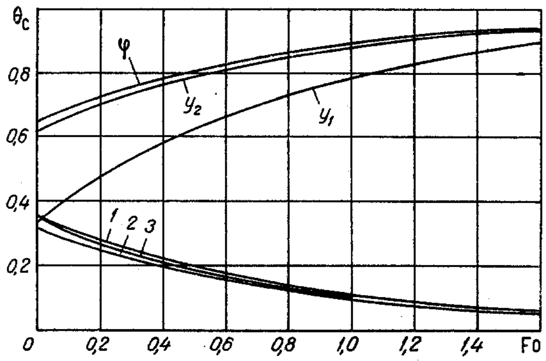

Fig. 1. Determination of the heat transfer coefficient: 1) ($\Theta_c - \varphi$); 2) ($\varphi - y_1$); 3) $\sum_{n=1}^{2}(\varphi - y_n)$.

Hence we obtain

$$\frac{\partial \Theta(X, \text{Fo})}{\partial X} \equiv \frac{\partial W(X, \text{Fo})}{\partial X} \qquad (A.6)$$

or, substituting X = 1,

$$\frac{\partial \Theta(1, \text{Fo})}{\partial X} \equiv \frac{\partial W(1, \text{Fo})}{\partial X}. \qquad (A.7)$$

This means that the derivative $\partial\Theta(1, \text{Fo})/\partial X$ and the function $\Theta(1, \text{Fo})$ are interrelated, not only by the boundary condition (2), but also by the relation determined by solution of the system (A.1)-(A.4) with boundary condition of the first kind (A.2). This relation is given by Eqs. (19), (20) and (21), where $\varphi(\text{Fo}) = \Theta(1, \text{Fo})$.

NOTATION

Θ	is the temperature of the plate material;
Θ_c	is the temperature of the medium;
x	is a spacial coordinate;
t	is time;
a	is the thermal diffusivity;
λ	is the thermal conductivity;
$\alpha(t)$	is the heat transfer coefficient;
L	is the plate thickness;
$X = x/L$	is a dimensionless coordinate;
$\text{Fo} = at/L^2$	is the Fourier number;
$\text{Bi(Fo)} = \alpha(\text{Fo})L/\lambda$	is the Biot number.

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